L^2-Betti numbers and non-unitarizable groups without free subgroups

D. V. Osin *

Abstract

We show that there exist non-unitarizable groups without non-abelian free subgroups. Both torsion and torsion free examples are constructed. As a by-product, we show that there exist finitely generated torsion groups with non-vanishing first L^2-Betti numbers. We also relate the well-known problem of whether every hyperbolic group is residually finite to an open question about approximation of L^2-Betti numbers.

1 Introduction

Let G be a group, H a Hilbert space. Recall that a representation $\pi\colon\ G \to B(H)$ is *unitarizable* if there exists an invertible operator $S\colon\ H \to H$ such that $g \to S^{-1}\pi(g)S$ is a unitary representation of G. A (locally compact) group G is *unitarizable* if every uniformly bounded representation $\pi\colon\ G \to B(H)$ is unitarizable.

In 1950, Day [3] and Dixmier [4] proved that every amenable group is unitarizable. The question of whether the converse holds has been open since then. A good survey of the current research in this direction is given in [19].

The simplest examples of a non-unitarizable groups are non-abelian free groups. (An explicit construction of a uniformly bounded non-unitarizable representation can be found [12]). Since unitarizability passes to subgroups, every group containing a non-abelian free subgroup is not unitarizable as well. However, the answer to the following question was unknown until now.

Problem 1.1. *Does there exist a non-unitarizable group without non-abelian free subgroup?*

Note that if there is such a group, it is a non-amenable group without non-abelian free subgroups. The existence of such groups remained a fundamental open problem for many years until the first examples were constructed by Olshanskii in [13]. The aim of this note is to answer the question affirmatively. Namely we prove the following.

Theorem 1.2. *There exists a finitely generated torsion non-unitarizable group.*

*This work has been supported by the NSF grant DMS-0605093

The proof of Theorem 1.2 is a combination of a sufficient condition for non-unitarizability found by Epstein and Monod [5], a recent result of Peterson and Thom [20] about first L^2-Betti numbers of groups defined by periodic relations, and some older techniques related to hyperbolic groups [6, 15]. Though the property of being torsion is crucial for this approach, we show that the torsion group from Theorem 1.2 can be used to construct examples of completely different nature.

Theorem 1.3. *There exists a finitely generated torsion free non-unitarizable group without free subgroups.*

As a by-product, we also obtain some new results about L^2-Betti numbers of groups. Recall that if G is a torsion free group satisfying the Atiyah Conjecture (or even a weaker property $(*)$ introduced in [20]), then $\beta_1^{(2)}(G) > 0$ implies the existence of non-abelian free subgroups in G [20, Theorem 4.1]. For finitely presented residually p-finite groups, where p is a prime, even a stronger result holds. Namely $\beta_1^{(2)}(G) > 0$ for such a group G implies that G is large [8]. These examples lead to a natural question of whether non-vanishing of the first L^2-Betti number always implies the existence of non-abelian free subgroups. The following theorem shows that the answer is negative.

Theorem 1.4. *There exists a finitely generated torsion group with non-vanishing first L^2-Betti number.*

Moreover, Theorem 1.4 allows us to relate a question about approximation of L^2-Betti numbers to one of the most intriguing open problems about hyperbolic groups. Recall that a group G is *residually finite* if for every element $g \neq 1$ of G there is a homomorphism $\varepsilon\colon G \to Q$, where Q is finite, such that $\varepsilon(g) \neq 1$. By the Approximation Theorem of Lück [9], for every residually finite finitely presented group G and every nested sequence of finite index normal subgroups $\{N_i\}$ of G with trivial intersection, one has

$$\beta_1^{(2)}(G) = \lim_{i\to\infty} \frac{b_1(N_i)}{[G:N_i]}, \tag{1}$$

where $b_1(N_i)$ is the ordinary first Betti number of N_i. The following question is still open.

Problem 1.5. *Does the approximation hold for any finitely generated residually finite group?*

The other problem is well-known. For a survey of the theory of hyperbolic groups we refer to [6, 2].

Problem 1.6. *Is every hyperbolic group residually finite?*

We show that if every hyperbolic group is residually finite, then the group from Theorem 1.4 can be made residually finite as well. However this contradicts (1) since $\beta_1(N_i) = 0$ for any subgroup of a torsion group. Thus we obtain the following.

Corollary 1.7. *At least one of the two problems has a negative solution.*

Acknowledgment I am grateful to Nicolas Monod for drawing my attention to the paper [5] and stimulating discussions. I am also grateful to Jesse Peterson for explaining results of [20].

2 Torsion groups with positive first L^2-betty numbers

Recall that a group is *elementary* if it contains a cyclic subgroup of finite index. For every hyperbolic group G and every element $g \in G$ of infinite order, there exists a (unique) maximal elementary subgroup $E(g) \leq G$ containing g (see, e.g., [15, Lemma 1.16].

Given an element g of a group G, we denote by $\langle\langle g \rangle\rangle$ the normal closure of g in G, i.e., the smallest normal subgroup of G containing g. Our main tool in this section is the following result about adding higher powered relations to hyperbolic groups. Up to little changes it was conjectured by Gromov [6]. It can easily be extracted from the proof of (a more complicated) Theorem 3 in [15]. Since the result we need does not formally follow from [15, Theorem 3], we briefly explain how to derive it from other results of [15] for convenience of the reader.

Lemma 2.1 (Olshanskii, [15])**.** *Let G be a hyperbolic group, S a finite subset of G, E a maximal elementary subgroup of G, C a finite index normal cyclic subgroup of E. Suppose that $C = \langle x \rangle$. Then for every sufficiently large integer n the following conditions hold.*

(1) The quotient group $G/\langle\langle x^n \rangle\rangle$ is hyperbolic.

(2) The image of the element x in $G/\langle\langle x^n \rangle\rangle$ has order n.

(3) The natural homomorphism $G \to G/\langle\langle x^n \rangle\rangle$ is injective on S.

Proof. Let X be a finite generating set of G, W a shortest word in $X \cup X^{-1}$ representing x in G. By [15, Lemma 4.1] the set of all cyclic shifts of the words $W^{\pm m}$ satisfies a small cancellation condition, which implies properties (1)-(3) for $G = \langle G \mid W^n = 1 \rangle \cong G/\langle\langle x^n \rangle\rangle$ by [15, Lemma 6.7] if $m = m(G, x, S)$ is sufficiently large. □

For a background on L^2-Betti numberst we refer the reader to [10]. In what follows we assume that $1/|G| = 0$ if a group G has infinite order. Recall that for $G = G_1 * \cdots * G_n$, we have

$$\beta_1^{(2)}(G) = n - 1 + \sum_{i=1}^{n} \left(\beta_1^{(2)}(G_i) - \frac{1}{|G_i|} \right) \tag{2}$$

(see[11]). The following theorem of Peterson and Thom [20] will allow us to control first L^2-Betti number after adding higher-powered relations to G.

Theorem 2.2. *[20, Theorem 3.2] Let G be an infinite countable discrete group. Assume that there exist subgroups $G_1, \ldots, G_n$ of G, such that*

$$G = \langle G_1, \ldots, G_n \mid r_1^{w_1}, \ldots, r_k^{w_k} \rangle,$$

for some elements $r_1, \ldots, r_k \in G_1 * \cdots * G_n$ and positive integers $w_1, \ldots, w_k$. Suppose in addition that the order of r_i in G is w_i. Then, the following inequality holds:

$$\beta_1^{(2)}(G) \geq n - 1 + \sum_{i=1}^{n} \left(\beta_1^{(2)}(G_i) - \frac{1}{|G_i|} \right) - \sum_{j=1}^{k} \frac{1}{w_j}.$$

Theorem 2.3. *For every positive integer n and every $\varepsilon > 0$, there exists an n-generated torsion group T with $\beta_1^{(2)}(T) \geq n - 1 - \varepsilon$. Moreover, if every hyperbolic group is residually finite, then the group T can be additionally made residually finite.*

Proof. Roughly speaking, the main idea of the proof is to start with the free product $G = \mathbb{Z}_m * \cdots * \mathbb{Z}_m$ and then add periodic relations $r^w = 1$ (one by one) for all $r \in G$, where m and all $w = w(r)$ are large enough. We are going to use Theorem 2.2 to prove that the first L^2-betti number of the groups obtained at each step is close to the number of free factors in G. A continuity argument will then help us carry over this estimate to the limit group. The only difficulty is to verify the hypothesis of Theorem 2.2 concerning orders of r_i's and to ensure that the images of $\mathbb{Z}_m$'s remain isomorphic to $\mathbb{Z}_m$ on each step. This is done by using hyperbolic groups and Lemma 2.1.

Let m be an integer such that $n/m < \varepsilon$. Let $G = G_1 * \cdots * G_n$, where $G_i \cong \mathbb{Z}_m$ for each $i = 1, \ldots, n$. We enumerate all elements of $G = \{1 = g_0, g_1, g_2, \ldots\}$, and construct the group T by induction. Let $T_0 = G$. Suppose that a group

$$T_i = \langle G_1, \ldots, G_n \mid r_1^{w_1}, \ldots, r_{k_i}^{w_{k_i}} \rangle,$$

$k_i \leq i$, has already been constructed for some $i \geq 0$. In what follows, we use the same notation for elements of T_0 and their images in T_i. We assume that:

(a) T_i is hyperbolic.

(b) The natural maps $G_l \to T_i$ are injective for $l = 1, \ldots, n$. In particular, we may think of G_l's as subgroups of T_i.

(c) $|r_j| = w_j$ in T_i for $j = 1, \ldots, k_i$.

(d) $\sum_{j=1}^{k_i} \frac{1}{w_j} + \frac{n}{m} < \varepsilon$.

(e) Elements $g_0, \ldots, g_i$ have finite orders in T_i.

Observe that the inductive assumption trivially holds for T_0. By Theorem 2.2, conditions (b), (c), and (d) imply that

$$\beta_1^{(2)}(T_i) > n - 1 - \varepsilon.$$

The group T_{i+1} is obtained from T_i in the following way. If the image of g_{i+1} has finite order in T_i, we set $k_{i+1} = k_i$ and $T_{i+1} = T_i$. Otherwise let C be an (infinite) finite index cyclic normal subgroup of $E(g_{i+1})$. Since $|E(g_{i+1})/C|$ is finite, there exists $m > 0$ such that $g_{i+1}^m \in C$. Consequently, $\langle g_{i+1}^m \rangle$ is normal in $E(g_{i+1})$. Passing to $\langle g_{i+1}^m \rangle$, we may assume that $C = \langle g_{i+1}^m \rangle$ without loss of generality. Let $k_{i+1} = k_i + 1$, $r_{k_{i+1}} = g_{i+1}$ and

$$S = \left(\bigcup_{l=1}^{n} G_l \right) \cup \left(\bigcup_{j=1}^{k_i} \langle r_j \rangle \right).$$

Applying Lemma 2.1 to the group $G = T_i$, the element $x = g_{i+1}^m$, and the subset S, we obtain that for every large enough integer s, the quotient group

$$T_{i+1} = \langle T_i \mid r_{k_{i+1}}^{sm} \rangle = \langle G_1, \dots, G_n \mid r_1^{w_1}, \dots, r_{k_i}^{w_{k_i}}, r_{k_{i+1}}^{sm} \rangle \tag{3}$$

is hyperbolic, $|g_{i+1}^m| = s$ (hence $|r_{k_{i+1}}| = |g_{i+1}| = ms$) in T_{i+1}, and the natural homomorphism $T_i \to T_{i+1}$ is injective on S. The later condition ensures that $|r_j| = w_j$ in T_{i+1} for $j = 1, \dots, k_i$ and the natural maps $G_l \to T_{i+1}$ remain injective for $l = 1, \dots, n$. By (d), we may choose s such that

$$\sum_{j=1}^{k_i} \frac{1}{w_j} + \frac{1}{ms} + \frac{n}{m} < \varepsilon.$$

Letting $k_{i+1} = k_i + 1$ and $w_{k_{i+1}} = ms$ completes the inductive step.

Let

$$T = \langle G_1, \dots, G_n \mid r_1^{w_1}, r_2^{w_2}, \dots \rangle$$

be the inductive limit of the groups T_i and the natural homomorphisms $T_i \to T_{i+1}$. By (e) the image of every element g_i has finite order in T, i.e., T is a torsion group.

Note that the sequence $\{T_i\}$ converges to T in the topology of marked group presentations. (For details about this topology we refer to [18].) Indeed this is always true whenever we have a sequence of normal subgroups $N_1 \leq N_2 \leq \dots$ of a group T_0, $T = T_0 / \bigcup_{i=1}^{\infty} N_i$, and $T_i = T_0/N_i$. (In our case N_i is the normal closure of $r_1, \dots, r_{k_i}$ in T_0). By semi-continuity of the first L^2-Betti number (see [18]), we obtain

$$\beta_1^{(2)}(T) \geq \lim_{i \to \infty} \beta_1^{(2)}(T_i) \geq n - 1 - \varepsilon.$$

Suppose now that every hyperbolic group is residually finite. Then we adjust our construction as follows. Let X be a finite generating set of T_0 and let d_i denote the word metric on T_i corresponding to the natural image of X in T_i. For every $i \in \mathbb{N}$, we choose a homomorphism $\tau_i \colon T_i \to Q_i$, where Q_i is finite, such that $\tau_i(t) \neq 1$ whenever $d_i(t, 1) \leq i$ and $t \neq 1$. Such a homomorphism always exists as T_i is hyperbolic and hence it is residually finite by our assumption.

Note that passing from T_i to T_{i+1} according to (3) we may always choose s to be a multiple of any given non-zero integer. Thus we may assume that w_j is divisible by $|Q_i|$ for any $i, j \in \mathbb{N}$, $j > i$. This implies that for every $i \in \mathbb{N}$, the kernel of the natural homomorphism $T_i \to T$ is contained in $\mathrm{Ker}(\tau_i)$ and hence τ_i factors through $T_i \to T$. Let σ_i be the corresponding homomorphism $T \to Q_i$.

Denote by d the word metric on T with respect to the natural image of the set X. If s is a nontrivial element of T such that $d(s, 1) = i$, then there is an element $t \in T_i$ such that $d_i(t, 1) \leq i$ and s is the natural image of t in T. According to our construction, $\tau_i(t) \neq 1$ and hence $\sigma_i(t) \neq 1$. Thus the group T is residually finite. $\square$

3 Non-unitarizable groups without free subgroups

Given a finitely generated group H, we denote by $\mathrm{rk}\,(H)$ its rank.

Theorem 3.1 (Epstein-Monod [5]). *Let G be a unitarizable group. Then the ratio $\beta_1^{(2)}(H)/\sqrt{\mathrm{rk}\,(H)}$ is uniformly bounded on the set of all finitely generated subgroups of G.*

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. By Theorem 2.3 for every integer $n \geq 2$, there exists a group G_n generated by n elements such that $\beta_1^{(2)}(G_n) \geq n-2$. Let G be the direct product of the family $\{G_n \mid n \geq 2\}$. Clearly G is a torsion group and is not unitarizable by Theorem 3.1. To complete the proof it remains to recall that every countable torsion group embeds into a torsion group generated by 2 elements [14], and every group containing a non-unitarizable subgroup is non-unitarizable itself. $\square$

To construct torsion free examples, we need another result about hyperbolic groups. Similarly to Lemma 2.1, it can be easily extracted from the proof of Theorem 2 in [15]. We make this extraction for convenience of the reader and refer to [15] for details and terminology.

Lemma 3.2 (Olshanskii, [15]). *Let G be a torsion free hyperbolic group, H a non-elementary subgroup of G, $t_1, \ldots, t_m$ elements of G. Then there exist elements $r_1, \ldots, r_m \in H$ such that the following conditions hold for the quotient group $G_1 = G/\langle\langle r_1t_1, \ldots, r_mt_m\rangle\rangle$.*

(1) G_1 is torsion free hyperbolic.

(2) The natural image of H is a non-elementary subgroup of G_1.

Observe that the images of the elements $t_1, \ldots, t_m$ belong to the image of H in G_1.

Proof. Since G is torsion free, all elementary subgroups of G are cyclic. Let g be any non-trivial element of H such that $E(g) = \langle g\rangle$. Let l be a positive integer, $x_1, \ldots, x_l \in H$ elements provided by [15, Lemma 3.7]. Let $W, X_0, X_1, \ldots, X_l$ be shortest words in a finite set of generators of G representing $g, r_1, x_1, \ldots, x_l$ respectively.

Using [15, Lemma 4.2] and triviality of finite subgroups in G, we obtain that the set of all cyclic shifts of the words $(X_0W^mX_1W^m \cdots X_lW^m)^{\pm 1}$ satisfies a small cancellation condition,which implies (1) and (2) for the group

$$G_1 = \langle G \mid X_0W^mX_1W^m \cdots X_lW^m = 1\rangle$$

by [15, Lemma 6.7] if m and l are large enough. Note that $G_1 \cong G/\langle\langle r_1t_1\rangle\rangle$, where t_1 is the element of H represented by $W^mX_1W^m \cdots X_lW^m$. Doing the same procedure for $r_2, \ldots, r_m$ we prove the lemma by induction. $\square$

Proof of Theorem 1.3. We are going to construct the desired group G as a (torsion free) extention $1 \to H \to G \to T \to 1$, where T is a non-unitarizable torsion group provided by Theorem 1.2 and H has no non-abelian free subgroups. It is easy to show that every such a group G is not unitarizable and does not contain non-abelian free subgroups.

More precisely, let

$$T = \langle x, y \mid r_1, r_2, \ldots \rangle, \tag{4}$$

be a presentation of a non-unitarizable torsion group. Without loss of generality we may assume T to be generated by 2 elements (see the proof of Theorem 1.2). Again we proceed by induction. Let $G_0 = \langle x, y, a, b\rangle$ be the free group with basis x, y, a, b. In what follows we construct a series of quotients of G_0. As in the proof of Theorem 2.3, we keep the same notation for elements of G_0 and their images in these quotient groups.

Clearly $H_0 = \langle a, b\rangle$ is a non-elementary subgroup of G_0. Hence by Lemma 3.2, there exist elements $u_1, \ldots, u_8, v_1 \in H$ such that the quotient group

$$G_1 = \langle G_0 \mid a^x u_1, a^{x^{-1}} u_2, b^x u_3, b^{x^{-1}} u_4, a^y u_5, a^{y^{-1}} u_6, b^y u_7, b^{y^{-1}} u_8, r_1 v_1 \rangle$$

is torsion free hyperbolic, and the image H_1 of H in G_1 is non-elementary. Without loss of generality we may assume that $u1, \ldots, u_8$ and v_1 are words in $\{a^{\pm 1}, b^{\pm 1}\}$.

We enumerate all finitely generated subgroups $H_1 = R_1, R_2, \ldots$ of H_1. Suppose that for some $i \geq 1$, we have already constructed a group

$$G_i = \left\langle a, b, x, y \;\middle|\; \begin{array}{c} a^x u_1, a^{x^{-1}} u_2, b^x u_3, b^{x^{-1}} u_4, a^y u_5, a^{y^{-1}} u_6, b^y u_7, b^{y^{-1}} u_8 \\ r_1 v_1, \ldots, r_i v_i \\ w_1, \ldots, w_{k_i} \end{array} \right\rangle$$

such that the following conditions hold. By H_i we denote the subgroup of G_i generated by a and b (i.e., the image of H_0 in G_i).

(a) The group G_i is torsion free hyperbolic.

(b) The subgroup $H_i = \langle a, b\rangle$ of G_i is non-elementary.

(c) $v_1, \ldots, v_i$ and $w_1, \ldots, w_{k_i}$ are words in $\{a^{\pm 1}, b^{\pm 1}\}$.

(d) For every $j = 1, \ldots, i$, the image of R_j in G_i is either cyclic or coincides with the image of H_i.

Clearly these conditions hold for $i = 1$. Relations $w_1, w_2, \ldots, w_{k_i}$ are absent in this case.

The group G_{i+1} is obtained from G_i in two steps. First, by parts (a), (b) of the inductive assumption and Lemma 3.2, we may choose a word v_{i+1} in $\{a^{\pm 1}, b^{\pm 1}\}$ such that the quotient group $K_i = G_i / \langle\!\langle r_{i+1} v_{i+1} \rangle\!\rangle$ is torsion free hyperbolic, and the natural image of H_i in K_i is non-elementary.

Further if the natural image of R_{i+1} in K_i is cyclic, we set $G_i = K_i$ and $k_{i+1} = k_i$. Otherwise the image of R_{i+1} in K_i is non-elementary. Indeed it is well-known and easy to prove that every torsion free elementary group is cyclic. Thus we can apply Lemma 3.2 to

the image of R_{i+1} in K_i and elements a, b. Let z_1, z_2 be elements of the image of R_{i+1} in K_i such that the quotient group $G_{i+1} = K_i / \langle\langle az_1, bz_2 \rangle\rangle$ is torsion free hyperbolic and the image of R_{i+1} in G_{i+1} is non-elementary. Recall that $R_{i+1} \leq H_1 = \langle a, b \rangle$. Hence we can assume that z_1, z_2 are words in $\{a^{\pm 1}, b^{\pm 1}\}$. Since $a = z_1^{-1}$ and $b = z_2^{-1}$ in G_{i+1}, the image of R_{i+1} in G_{i+1} coincides with the subgroup $H_{i+1} = \langle a, b \rangle$ of G_{i+1}. In particular, H_{i+1} is non-elementary. Note that az_1, bz_2 are words in $\{a^{\pm 1}, b^{\pm 1}\}$ as well. Let $w_{k_i+1} = az_1$, $w_{k_i+2} = bz_2$. The inductive step is completed.

Let now G be the inductive limit of the sequence $G_0, G_1, \ldots$. That is,

$$G_i = \left\langle a, b, x, y \;\middle|\; \begin{array}{c} a^x u_1, a^{x^{-1}} u_2, b^x u_3, b^{x^{-1}} u_4, a^y u_5, a^{y^{-1}} u_6, b^y u_7, b^{y^{-1}} u_8 \\ r_1 v_1, r_2 v_2, \ldots \\ w_1, w_2, \ldots \end{array} \right\rangle \tag{5}$$

Let also $H = \langle a, b \rangle$ be the natural image of H_1 in G. Note that the elements $a^x, a^{x^{-1}}, b^x, b^{x^{-1}}, a^y, a^{y^{-1}}, b^y, b^{y^{-1}}$ belong to H in G, as $u_1, \ldots, u_8$ are words in $\{a^{\pm 1}, b^{\pm 1}\}$. Hence H is a normal subgroup of G. Obviously $G/H \cong T$. Indeed after imposing additional relations $a = 1$ and $b = 1$, the relations corresponding to the first and the third rows of (5) disappear and the second row of (5) becomes $r_1, r_2, \ldots$ (see (c)). After removing a, b from the set of generators, we obtain exactly the presentation (4) of T.

Thus the group G splits as $1 \to H \to G \to T \to 1$. Note that every finitely generated proper subgroup of H is cyclic. Indeed, let Q be a finitely generated subgroup of H, P some finitely generated preimage of Q in H_1. Then $P = R_i$ for some i. The natural homomorphism $P \to Q$ obviously factors through the image of $P = R_i$ in G_i. However the image of R_i in G_i is either cyclic or coincides with H_i by (d). Therefore, we obtain that Q is either cyclic or coincides with H.

Let F be a nontrivial finitely generated free subgroup of G. Then $F \cap H$ is cyclic. Note also that $F \cap H$ is normal in F and $F \cap H \neq 1$ since $F/(F \cap H) \cong FH/H \leq T$ is a torsion group. Thus F has a nontrivial normal cyclic subgroup, and hence F is cyclic itself. This shows that G contains no non-abelian free subgroups. Further suppose that some element $g \neq 1$ has finite order in G. This means that for some $n > 0$, the relation $g^n = 1$ follows from relations of the presentation (5). Hence it follows from some finite set of relations of (5), i.e., $g^n = 1$ holds in G_i for some i contrary to (a). Finally we note that G is not unitarizable since it surjects onto a non-unitarizable group T. $\square$

Remark 3.3. Let G be finitely generated group, X a finite generating set of G, $\varkappa(G, l^2(G), X)$ the Kazhdan constant of the left regular representation λ_G with respect to X. More precisely, $\varkappa(G, l^2(G), X)$ is defined as the supremum of all $\varepsilon > 0$, such that for every vector $u \in l^2(G)$ of norm 1, there exists $x \in X$ such that $\|\lambda_G(x)u - u\| \geq \varepsilon$. Recall that a finitely generated group G is *amenable* if and only is $\varkappa(G, l^2(G), X) = 0$ for every finite generating set X of G [7].

Let

$$\alpha(G) = \inf_{\langle X \rangle = G} \varkappa(G, l^2(G), X) = 0, \tag{6}$$

where the infimum is taken over all finite generating sets X of G. A finitely generated group G is called *weakly amenable* if $\alpha(G) = 0$. (A similar notion of weak amenability was

considered in [1].) Clearly every amenable group is weakly amenable. The converse was shown to be wrong in [16].

Combining methods of [16] and [17], one can construct a (finitely generated) non-unitarizable torsion group T without free subgroup such that every non-elementary hyperbolic group surjects onto T. In particular, every such a group is weakly amenable [16]. This shows that the absence of free subgroups does not imply unitarizability even being combined with weak amenability.

References

[1] G. Arzhantseva, J. Burillo, M. Lustig, L. Reeves, H. Short, E. Ventura, Uniform non-amenability, *Adv. Math.* **197** (2005), no. 2, 499–522.

[2] M. Bridson, A. Haefliger, Metric spaces of non–positive curvature, Springer, 1999.

[3] M. Day, Means for the bounded functionsand ergodicity of the bounded representations of semi-groups, *Trans. Amer. Math. Soc.* **69** (1950), 276–291.

[4] J. Dixmier, Les moyennes invariantes dans les semi-groups et leurs applications. (French) *Acta Sci. Math. Szeged* **12** (1950). Leopoldo Fejer et Frederico Riesz LXX annos natis dedicatus, Pars A, 213–227.

[5] I. Epstein, N. Monod, A note on non-unitarisable representations and random forests, arxiv:arXiv:0811.3422.

[6] M. Gromov, Hyperbolic groups, Essays in Group Theory, MSRI Series, Vol.8, (S.M. Gersten, ed.), Springer, 1987, 75–263.

[7] A. Hulanicki, Groups whose regular representation weakly contains all unitary representations, *Studia Math.* **24** (1964), 37–59.

[8] M. Lackenby, Detecting large groups, arXiv:math/0702571.

[9] W. Lück, Approximating L^2-invariants by their finite-dimensional analogues, *Geom. Funct. Anal.* **4** (1994), no. 4, 455–481.

[10] W. Lück, L^2-invariants: theory and applications to geometry and K-theory. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 44. Springer-Verlag, Berlin, 2002.

[11] W. Lück, L^2-Betti numbers of some amalgamated free products of groups. Appendix to the paper "Free entropy dimension in amalgamated free products" by N.P. Brown, K.J. Dykema, K. Jung, *Proc. London Math. Soc.* **97** (2008), no. 2, 339–367.

[12] A.M. Mantero and A. Zappa, Uniformly bounded representations and Lp-convolution operators on a free group, Harmonic analysis (Cortona, 1982), 333343, Lecture Notes in Math., 992, Springer, Berlin, 1983.

[13] A.Yu. Olshanskii, On the question of the existence of an invariant mean on a group (Russian), Uspekhi Mat. Nauk **35** (1980), no. 4, 199–200.

[14] A.Yu. Ol'shanskii, Embedding of countable periodic groups in simple 2-generator periodic groups, *Ukrainian Math. J.* **43** (1991), no. 7-8, 914–919.

[15] A.Yu. Olshanskii, On residualing homomorphisms and G–subgroups of hyperbolic groups, *Int. J. Alg. Comp.*, **3** (1993), 4, 365–409.

[16] D. Osin, Weakly amenable groups, *Contemp. Math.* **298** (2002), 105-113.

[17] D. Osin, Small cancellations over relatively hyperbolic groups and embedding theorems, arXiv: math/0411039.

[18] M. Pichot, Semi-continuity of the first l^2-Betti number on the space of finitely generated groups, *Comment. Math. Helv.* **81** (2006), no. 3, 643–652.

[19] G. Pisier, Are unitarizable groups amenable? In *Infinite groups: geometric, combinatorial and dynamical aspects*, Progr. Math., **248**, 323–362, Birkhäuser, Basel, 2005.

[20] A. Thom, J. Peterson, Group cocycles and the ring of affiliated operators, arXiv:0708.4327.